Abstract

The issue of the optimality of time-diversification is closely linked to the equity premium puzzle. In this paper, we address this question in an economy of loss averse investors. Using piecewise-linear utility functions, we get analytical results thanks to option pricing theory. We show that loss aversion alone cannot explain the equity premium. Benartzi and Thaler (1995), by means of simulations on US data, evaluated the evaluation period, that is the implicit horizon of agents. We obtain similar results by analytical methods.

Classification JEL : G12, D81

Is time-diversification efficient for a loss averse investor ?

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1 Introduction

The issue of the efficiency of time-diversification is an important question for fund managers and economic agents as well. It can be closely related to several theoretical approaches in decision science and financial economics. Moreover, historical data show that stocks have consistently overperformed (relative to bonds or less risky assets) when sufficiently long holding periods are considered. This empirical fact cannot be justified by risk aversion in the usual expected utility framework. In their celebrated study of the equity premium, Mehra and Prescott (1985) have shown that the risk premium on stocks may be justified only if agents display an incredibly high coefficient of relative risk aversion (over 30 when EU theory usually predicts a coefficient in the neighborhood of 1).

The first approach of this problem may be traced back to Samuelson's paper (1963) in which it is proved that an agent rejecting a gamble X at all wealth levels will also reject any sequence of i.i.d gambles $(X_i, i = 1, .., N)$ when X and the X_i's have the same distribution. Ross (1999), coming back on Samuelson's result, has argued that the only utility functions for which gambles may be rejected at all wealth levels are the linear and the negative exponential utility functions; many other functions may accept sufficiently long sequences of individually undesirable bets. Moreover, experimental results

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(Bernartzi and Thaler, 1999) show that the willingness to play 100 Samuelson's bets depends on several factors, especially on the stake in each bet and on the way the problem is presented. When players are shown the final distribution of the results, the percentage of agents ready to play is greater than when they only know the number of bets to be performed and the probability distribution of each bet. It suggests misperceptions in probabilities, at least for rare events. In the portfolio problem, these results are interpreted as follows; if successive stock returns are independent random variables, an agent may be reluctant to invest in stocks in the short run; however, if his investment horizon is long, for example for retirement savings, he may be ready to invest massively in stocks.

Though it is an easy analogy, the question of time-diversification in portfolio management is more complicated, because, as the name clearly indicates, time is explicitly present. It is not only a sequential decision process, as in repeated bets, but also a sequence of investment decisions that have to be taken at different moments of the agent's lifetime. Moreover agents may dynamically revise the composition of their portfolio. Consequently, the historical distribution of stock returns may not be a sufficient argument to convince agents to invest massively in stocks, even if they have a long horizon; this is especially the case if they can observe drastic price drops in some periods.

Merton (1969) and Samuelson (1969) have shown that the proportion invested in stocks does not depend on the horizon for power utility functions. But if labor income is taken into account, a positive relationship between horizon and percentage invested in stocks may appear (Viceira, 2001 or Campbell and Viceira, 2002, chapters 6 and 7).

One of the answers given in the recent literature is to consider that agents exhibit myopic loss aversion, combining loss aversion with mental accounting. Loss aversion means that gains and losses do not receive the same weight in the agent's valuation function, losses being more weighted than gains ; this is one of the key features of prospect theory (Kahneman-Tversky, 1979, Tversky and Kahneman, 1992) . Mental accounting means that agents value their portfolios at regular (possibly short) intervals and that their welfare is greatly influenced by their intermediate valuations, even if they have a long horizon. Hence, as historical data show a high probability of large stock price drops (at least in the short run), the historical equity premium may be justified by such a myopic behavior. For example, Bernartzi and Thaler (1995) have justified the long run equity premium on stocks by an evaluation period[1] of about one year. Groot and Dijkstra (1996) have found even shorter evaluation periods in four developed countries. Nevertheless, they have observed a high variability in evaluation periods, across countries and through time.

It must be pointed out that gains and losses can only be defined with respect to a reference level which may vary through time. Barberis *et al.* (2001) consider a time-varying reference level to take into account the risk-free interest rate and the "house money"[2] effect as well, meaning that a loss is less painful if it is borne after preceding gains. An alternative interpretation is that the discounting factor used by an agent depends on the history of her portfolio values and an unanswered question concerns the weight to apply to this history.

In the financial literature, a rather extreme approach to define risk can be found, considering only downside risk. Bodie (1995) has measured risk as the price of a put option which, added to the portfolio of stocks, prevents the investor's wealth to end below a specified target level. It can be considered as an insurance approach : how much agents are ready to pay to be protected against low returns ?

Obviously, the put price depends on the quality of the hedge it generates. Bodie has considered that the reference level is the initial wealth capitalized at the risk-free rate. Under this assumption, the put price is an increasing function of the horizon, leading the author to the same conclusion as Samuelson, that is to say, time-diversification is a fallacy. Soon after, Merrill-Thorley (1996) and Zou (1997) have argued that Bodie's result was due to the arbitrary choice of the reference wealth level ; considering lower levels of terminal wealths, they have obtained a non-monotonic put price (with respect to the horizon of investment).

In this paper, we want to mix the two approaches, prospect theory and option pricing ; we first consider a loss averse investor endowed with a piecewise linear utility function ; this allows us to formulate expected utility as the value of a portfolio of call and put options[3]. However, we value these options under the real probability distribution, not under the risk-neutral one usually entering financial models. In this simple context, the loss aversion coefficient

[1] Delay between two evaluations.

[2] also called the "two pocket gambler" phenomenon.

[3] Groot and Dijsktra (1996) briefly analyze the same assumption in their appendix 3 and Bernartzi and Thaler mention a loss aversion coefficient of 2.77 (evaluated through simulations) for such a utility function

(greater than 1) is simply interpreted as the number of put options sold by the agent. A piecewise linear utility function being kinked at the reference level, it may also explain the fact that many economic agents do not possess any stock in their portfolio[4], especially if their investment horizon is short.

We define a critical investor, that is, an agent characterized by a loss aversion parameter which keeps him indifferent between investing in stocks and investing in the risk-free asset. We then study the relationship between this "-critical" parameter and the horizon of investment and show that this relationship is increasing. This result is an argument in favor of time-diversification even if the reference wealth level is chosen as the initial wealth capitalized at the risk-free rate as in the option-based approach. The conclusion is obviously reinforced if a lower level is chosen for the reference wealth level. This analysis is first performed in an illustrative way in a two-period model and then generalized in continuous-time.

We then consider the problem of mental accounting ; following Bernartzi and Thaler, we determine the evaluation period justifying the historical equity premium, using the value of the loss aversion parameter they have estimated by means of simulations. We then illustrate two other approaches[5] based, first on horizon-varying discounting factors implicit in the equity premium and second on a term structure of volatilities. This analysis can be linked to Barberis (2000) results on portfolio management with uncertainty on the parameters.

2 Loss aversion in a discrete-time model

2.1 General formulation in a one-period model

We first consider a simple one-period model with loss averse investors endowed with utility functions defined as follows[6] :

$$v(x) = v_G(x - x^*)\mathbf{1}_{\{x-x^* \geq 0\}} + v_L(x - x^*)\mathbf{1}_{\{x-x^* < 0\}}$$

[4]De Bondt (1998) reports that only 28% of american households were holding stocks in 1992, the percentage being even lower in Europe.

[5]This approach has been used by Dupré and Louvet (2000) to reinterpret the CAPM relationship in the context of time-diversification.

[6]The notations are those of Gomez (2000).

where indices G and L refer to gains and losses while x^* defines the reference level. The usual assumptions on the functions v_G and v_L are the following :

$$\begin{aligned} 1)\ v_G' &> 0; v_G'' \leq 0 \qquad (1) \\ 2)\ v_L' &> 0; v_L'' \geq 0 \\ 3)\ \frac{v_L'(0)}{v_G'(0)} &> 1 \\ 4)\ v_G(0) &= v_L(0) = 0 \qquad (2) \end{aligned}$$

The first two conditions imply that the agent is risk-averse in the domain of gains and risk-lover in the domain of losses. The last condition specifies the continuity of the utility function. Condition (3) defines the index of loss aversion proposed by Köbberling and Wakker (2002).

In this paper, we use the most simple formulation of such a utility function by assuming that v is piecewise linear as follows :

$$v(x) = \begin{cases} (x - x^*) \text{ if } x \geq 0 \\ \lambda (x - x^*) \text{ if } x < 0 \end{cases}$$

The assumption $\lambda > 1$ takes into account loss aversion and corresponds to the third condition in (1).

Consider now a financial market with two assets traded at date 0 : first, there is a risky asset (referred to as "the stock" in the following), the random terminal payment of which, denoted as X, taking N values $(x_1, ..., x_N)$ with probabilities $(\pi_1, ..., \pi_N)$; second, a risk-free asset (called the bond) paying a rate of return denoted as $r - 1$. The initial price of the bond is normalized to 1. An investor endowed with an initial wealth W_0 will be ready to buy stocks if his expected utility after the transaction is not lower than before. We consider, for the moment, that the date-1 reference level is rW_0, the terminal wealth level for a 100% investment in the risk-free asset.

Given the utility function and the chosen reference level, we assume without loss of generality that $W_0 = 0$; an investor who wants to buy the stock borrows the initial price S_0 at the risk-free rate and gets a net cash-flow $S_1 - rS_0$ at the terminal date. In other words, purchasing the risky stock is desirable only if the following inequality is verified :

$$\sum_{i=1}^{N} \pi_i \left((x_i - rS_0) \mathbf{1}_{\{x_i - rS_0 \geq 0\}} + \lambda (x_i - rS_0) \mathbf{1}_{\{x_i - rS_0 < 0\}} \right) \geq 0$$

Consequently, the assumption $W_0 = 0$ is neutral as the utility generated by investing in the risk-free asset is equal to 0.

It is then convenient to assume (without loss of generality, as long as there is only one risky investment) that the x_i are ranked in increasing order. We can then rewrite the expected utility as :

$$E\left[v(X)\right] = \lambda \sum_{i=1}^{i_0-1} \pi_i(x_i - rS_0) + \sum_{i=i_0}^{N} \pi_i(x_i - rS_0) \tag{3}$$

where $i_0 = \inf \{i \in \{1; ...; N\} / x_i \geq rS_0\}$.

As in the following sections we need to compare expected utilities for different horizons, we have to consider functions ν_T corresponding to the horizon T. We define, for a portfolio paying X_T at date T, the expected utility associated to T, by :

$$E\left[v_T(X_T)\right] = \frac{1}{r^T}\left[\lambda \sum_{i=1}^{i_0-1} \pi_i(x_i - rS_0) + \sum_{i=i_0}^{N} \pi_i(x_i - rS_0)\right]$$

The first question we want to address is to define the loss aversion parameter which makes the corresponding investor indifferent between investing in the stock or in the risk-free asset. The second point is to value a put option for such an investor when this financial security is used as a hedge. As this section is essentially illustrative, we will restrict the model to a binomial tree.

2.2 The critical loss aversion index in the two-states, one period model

Assume that the date-1 payment of the risky security takes only two values S^u and S^d ($S^u > S^d$) with probabilities p and $1-p$. We simply get the following preliminary result.

Proposition 1 *The loss aversion parameter λ, keeping the investor indifferent between the two assets, is defined by $\lambda^* = \frac{p}{1-p}\frac{u-r}{r-d}$ where u and d are defined by*[7] *$S^u = uS_0$ and $S^d = dS_0$.*

[7] These are the classical notations of Cox-Ross-Rubinstein (1979).

Proof. It is well known that the absence of arbitrage opportunities implies $d < r < u$. Hence, indifference between the stock and the risk-free asset means $E(v_1(S_1)) = 0$ where S_1 stands for the date-1 stock price. We can then write :

$$\frac{1}{r}\left[p(S^u - rS_0) + \lambda^*(1-p)\left(S^d - rS_0\right)\right] = 0 \tag{4}$$

Isolating S_0 gives :

$$S_0 = \frac{1}{r}\frac{pS^u + \lambda^*(1-p)S_d}{p + \lambda^*(1-p)}$$

Replacing S^u and S^d by their values give the desired result, that is :

$$\lambda^* = \frac{p}{1-p}\frac{u-r}{r-d} \tag{5}$$

■

An investor endowed with a loss aversion parameter equal to λ^*, as defined in equation (5) will be called a critical one-period investor. It must be noticed that discounting the utility function does not change the value of the critical λ because gains and losses are discounted at the same rate.

As a corollary, consider the price the critical investor is ready to pay for a put option preventing his final wealth to become negative, that is, a put option with strike price $K = S_0 r$. We obtain the following rather intuitive result.

Corollary 2 *When $\lambda^* = \frac{p}{1-p}\frac{u-r}{r-d}$, the reservation price of the put, denoted as ϕ_0, is equal to the arbitrage-free price, defined by*[8] *:*

$$\phi_0 = \frac{1}{r}(r-d)\,S_0\left(\frac{u-r}{u-d}\right)$$

Proof.

The agent borrows $S_0 + \phi_0$ at time 0, the put being devoted to protect the risky portfolio against returns lower than r; writing the expected utility gives the following relationship :

$$p\left(uS_0 - (S_0 + \phi_0)\,r\right) - \lambda(1-p)\phi_0 r = 0 \tag{6}$$

[8] This formula is deduced from the classical Cox-Ross-Rubinstein (1979) model.

The first term in equation 6 is the value of the terminal payoff in case of an up-state. If the price goes to dS_0, the agent exercizes the put option and gets rS_0. As he must reimburse $(S_0 + \phi_0)\, r$, the net cash-flow is equal to $-\phi_0 r$.

Isolating ϕ_0 and replacing λ by its value gives immediately the result :

$$\begin{aligned}\phi_0 &= \frac{p\,(u-r)\,S_0}{r(p+\lambda(1-p))} \\ &= \frac{1}{r}\frac{(u-r)\,S_0}{(1+\left(\frac{u-r}{r-d}\right))} = \frac{1}{r}\left(\frac{u-r}{u-d}\right)(r-d)\,S_0\end{aligned}$$

■

In such a simple model, the role of the loss aversion index is clearly identified and allows to get directly the link between the risk-neutral probability (appearing in the option valuation formula) and the physical probability. It is worth noting that the formulation of the put price does not depend on the physical probability of a down-state. The relationship between the two numbers is implicit because λ^* has been chosen in such a way that the risk in the underlying stock is exactly compensated by the expected return, for the critical investor under consideration.

2.3 The two-periods model

Suppose now that the investor can buy the stock for two periods, the per-period returns being i.i.d with the same distribution as in the preceding section. The following proposition shows that the investor characterized by $\lambda = \frac{p}{1-p}\frac{u-r}{r-d}$ is ready to buy the stock, then reaching a strictly positive expected utility level. In the following proposition, S_2 denotes the random date-2 value of the stock. In other words, this proposition indicates that the critical two-periods investor is less loss averse. This approach will be generalized in the next section to the continuous time-case.

Proposition 3 *If $\lambda = \frac{p}{1-p}\frac{u-r}{r-d}$, the expected utility of the agent, $E(v_2(S_2))$ is strictly positive.*

Proof

S_2 takes three values u^2S_0, udS_0 and d^2S_0 with respective probabilities $p^2, 2p(1-p), (1-p)^2$. We assume without loss of generality that $S_0 = 1$. Two cases have to be distinguished, depending on the sign of $ud - r^2$.

1) $ud > r^2$

$$r^2 E(v_2(S_2)) = p^2\left(u^2 - r^2\right) + 2p(1-p)\left(ud - r^2\right) + \lambda(1-p)^2\left(d^2 - r^2\right)$$

Replacing λ by its value gives :

$$r^2 E(v(S_2)) = p^2\left(u^2 - r^2\right) + 2p(1-p)\left(ud - r^2\right) + p(1-p)\left(d^2 - r^2\right)\frac{u-r}{r-d}$$

As we are only interested in the sign of $E(v(S_2))$, we note $A = \frac{r^2 E(v(S_2))}{p}$.

$$\begin{aligned}
A &= p\left(u^2 - r^2\right) + 2(1-p)\left(ud - r^2\right) - (1-p)(r+d)(u-r) \\
&= p\left(u^2 - r^2\right) + (1-p)\left(2\left(ud - r^2\right) - (r+d)(u-r)\right) \\
&= p\left(u^2 - r^2\right) + (1-p)\left(ud - r^2 - ru + rd\right) \\
&= p\left(u^2 - r^2\right) + (1-p)\left(u(d-r) - r(r-d)\right) \\
&= p\left(u^2 - r^2\right) + (1-p)\left((u+r)(d-r)\right) \\
&= (u+r)\left(p(u-r) + (1-p)(d-r)\right)
\end{aligned}$$

The investor being risk-averse, the second term of the product is then strictly positive because $\lambda > 1$.

2) $ud < r^2$

$$r^2 E(v_2(S_2)) = p^2\left(u^2 - r^2\right) + \lambda\left(2p(1-p)\left(ud - r^2\right) + (1-p)^2\left(d^2 - r^2\right)\right)$$

Using the same notation as in case (1), and replacing λ by its value gives :

$$\begin{aligned}
A &= p\left(u^2 - r^2\right) + \frac{u-r}{r-d}\left(2p\left(ud - r^2\right) + (1-p)\left(d^2 - r^2\right)\right) \\
&= p\frac{u-r}{r-d}\left[p(u+r)(r-d) + 2p\left(ud - r^2\right) + (1-p)\left(d^2 - r^2\right)\right]
\end{aligned}$$

Developing the term between brackets leads to :

$$\begin{aligned}
A &= p\frac{u-r}{r-d}\left[p(ur - rd + ud - d^2) + \left(d^2 - r^2\right)\right] \\
&= p\frac{u-r}{r-d}(r+d)\left[p(u-d) + (d-r)\right]
\end{aligned}$$

It remains to use the assumption $\lambda > 1$:

$$\lambda = \frac{p(u-r)}{(1-p)(r-d)} = \frac{p(u-d)}{(1-p)(r-d)} - \frac{p}{(1-p)} > 1$$

From this equality we deduce :

$$\frac{p(u-d)}{(1-p)(r-d)} > 1 + \frac{p}{(1-p)} = \frac{1}{1-p}$$

This in turn implies that $\frac{p(u-d)}{(r-d)} > 1$ meaning that $p(u-d)+(d-r) > 0$. ■

This result can be interpreted in several ways; first, if we assume that investors are characterized by the same λ, investors with a short horizon will sell their stocks to investors with longer horizons. We cannot be satisfied by this idea because it leads to a quite surprising result; the only investors possessing stocks are the ones with the longest horizon. The alternative explanation, given by Bernartzi and Thaler (1995) is that, even long-term investors (two periods in our case) value their portfolios at date 1 and their overall welfare depends on gains and losses at this date (they exhibit myopic loss aversion).

2.4 Some alternative interpretations of the equity premium

In this section, we examine briefly alternative explanations of the equity premium in a simplified way; more precisely, we consider that the discounting rate or the volatility of returns, used by the agent, may be period dependent implying indifference between a one-period and a two-periods investment. We illustrate the fact that an increasing risk-free rate (or adjustment rate of the reference wealth level) or an increasing volatility may lead the agent to be indifferent between two horizons.

2.4.1 The term structure of interest rates

An other way to explain the equity premium is to consider a richer dynamics for the reference wealth level, assuming that the date-2 level is equal to W_0rf where f is equal to one plus the capitalization rate used by the agent in period 2. Notation f is justified by the nature of this interest rate which is in fact a forward rate when considered at date 0. Noting that $E(v_2(S_2) > 0$ implies that the forward rate, which gives a two-period expected utility equal to 0, is greater than r. In fact, we could solve :

$$0 = p^2\left(u^2 - rf\right) + 2p(1-p)\left(ud - rf\right) + \lambda(1-p)^2\left(d^2 - rf\right)$$

where

$$\lambda = \frac{p}{1-p}\frac{u-r}{r-d}$$

Consider the following parameters :

$$p = 0.5; u = 1.3; d = 0.9; r = 1.03$$

We get :

$$(1.69 - 1.03f) + 2(1.17 - 1.03f) + \frac{0.27}{0.13}(0.81 - 1.03f) = 0$$

leading to the solution , $f = 1.0924$, that is a forward rate equal to 9.24%. It implies a two-year rate equal to $R = \sqrt{1.0924 \times 1.03} - 1 = 6.07\%$. This approach will be developed in the following section.

2.4.2 The term structure of volatilities

An other important way to interpret the preceding result relies on volatility. In the two-period model, the volatility of returns is assumed to remain the same in each period. However, it can be seen on the value of λ that the critical one-period parameter is an increasing function of volatility; roughly speaking, a higher volatility implies a higher u and a lower d implying a greater value of λ. This argument may also be applied to the parameters of the second period. One way to obtain $E(v_2(S_2)) = 0$ is to consider a greater volatility in the second period. This interpretation is in the spirit of Barberis (2000) paper in which the estimation risk is examined, that is the risk about parameters of the asset valuation model to be used by the investors. This point will be examined in more details in the continuous-time model.

3 The continuous-time model

3.1 The dynamics of the stock price

We consider now the continuous-time case and show that the loss aversion parameter of the critical investor is increasing as a function of the investment horizon. We assume that the dynamics of the risky asset price is governed by a geometric brownian motion characterized by :

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$$

where Z denotes a standard brownian motion, μ is the drift parameter and σ the diffusion coefficient, that is to say, the volatility of returns. The date-T price of the stock is then defined by :

$$S_T = S_0 \exp\left((\mu - \frac{\sigma^2}{2})T + \sigma Z_T \right)$$

The dynamics of the risk-free asset is simply :

$$dB_t = rB_t dt$$

where r is the risk-free rate, assumed to be lower than μ and constant through time. This formulation is the usual one appearing in the financial literature, especially in the Black-Scholes model (1973). It must be noted that r is now the continuously compounded risk-free interest rate ; in the discrete-time model, it was equal to one plus the one-period risk-free rate.

3.2 The issue of time-diversification

One of the essential arguments concerning time-diversification refers, in a more or less sophisticated manner, to the law of large numbers. Successive returns being independent, the volatility of returns evolves as the square root of time when expected return is a linear function of time. When the horizon of investment lenghtens, investing in stocks appears to be less risky. These arguments have been widely debated in the literature; taking into account loss aversion in place of risk aversion is a different way to approach the problem. In fact, for long horizons, the probability of losses diminishes but maximum possible losses increase. Consequently, what happens is not clear for a loss averse investor. To deal with this problem in a general framework, we analyze how evolves the critical loss aversion parameter with respect to the horizon of investment, denoted as T. To be more precise concerning notations, we will denote as λ_T this parameter and by Δ_T the event

$$\Delta_T = \{S_T - S_0 \exp(rT) > 0\} = \{S_T \exp(-rT) - S_0 > 0\} \tag{7}$$

$$= \left\{ Z_T > -\frac{(\mu - r - \frac{\sigma^2}{2})T}{\sigma} \right\} \tag{8}$$

where $S_0 \exp(rT)$ is the reference level for the investor under consideration. The choice of r for the capitalization rate could be considered as arbitrary but, as we will show later on, an other rate would lead to the same results as long as this rate is lower than μ.

The expected utility of the critical investor is, by definition, equal to 0 and is written as :

$$\begin{aligned} E\left(v_T(S_T)\right) &= \exp(-rT)E\left[(S_T - S_0\exp(rT))\mathbf{1}_{\Delta_T} + \lambda_T(S_T - S_0\exp(rT))\mathbf{1}_{\Delta_T^c}\right] = 0 \\ &= E\left[(S_T\exp(-rT) - S_0)\mathbf{1}_{\Delta_T} + \lambda_T(S_T\exp(-rT) - S_0)\mathbf{1}_{\Delta_T^c}\right] \end{aligned}$$

where Δ_T^c stands for the complementary set of Δ_T. We then deduce :

$$\lambda_T = \frac{E\left[\max(S_T\exp(-rT) - S_0; 0)\right]}{E\left[\max(S_0 - S_T\exp(-rT); 0\right]}$$

This equality comes directly from the definition of Δ_T and can be rewritten as :

$$\lambda_T = \frac{E\left[\max(S_T^* - S_0; 0)\right]}{E\left[\max(S_0 - S_T^*; 0\right]}$$

with

$$S_T^* = S_0\exp\left((\mu - r - \frac{\sigma^2}{2})T + \sigma Z_T\right)$$

It then appears that the numerator is the expected terminal payment of a call with strike price S_0 and the denominator is the expected terminal payment of the corresponding put. However, the discounted stock price follows a geometric brownian motion with parameters $\alpha = \mu - r$ and σ. It follows that the numerator of λ_T is an increasing function of the corresponding option lifetime, that is T, as long as the expected return on the stock is greater than the risk-free rate. The evolution of the denominator is not so simple and we prove in the following that it is non-monotonic[9] with respect to T.

Proposition 4 *Let $\lambda > 1$ fixed ; there exists an horizon T^* such that an investor characterized by λ will invest in stocks if his horizon is greater than T^*.*

[9] The same argument is used by Merrill-Thorley (1996) and Zou (1997) to refute Bodie's result on time-diversification.

Proof : It is worth to notice that the condition $\alpha > 0$ is natural in a world with risk-averse (or loss-averse) agents. We have to prove that expected utility becomes positive beyond an horizon T^*. Let us write

$$\begin{aligned} C_T &= E\left[\max(S_T^* - S_0; 0)\right] \\ P_T &= E\left[\max(S_0 - S_T^*; 0)\right] \end{aligned}$$

We can now use the Black-Scholes formula to value the first term in a simplified manner because the strike price is now equal to S_0. After simplifications, we obtain :

$$E\left[\max\left(S_T^* - S_0; 0\right)\right] = S_0 \exp(\alpha T) \left(N(d_1^T) - \exp\left(-\alpha T\right) N(d_2^T)\right)$$

where $d_1^T = \frac{\left(\alpha + \frac{\sigma^2}{2}\right)\sqrt{T}}{\sigma}$ and $d_2^T = d_1^T - \sigma\sqrt{T}$. This equality results directly from the brownian assumption. However it is different from the usual Black-Scholes formulation because expectations are evaluated under the real probability, not the risk-neutral one.

The same transformations applied to the denominator give :

$$P_T = S_0 \exp(\alpha T) \left(\exp\left(-\alpha T\right) N(-d_2^T) - N(-d_1^T)\right)$$

We then obtain the critical λ_T using the following formulation :

$$\lambda_T = \frac{\left(N(d_1^T) - \exp\left(-\alpha T\right) N(d_2^T)\right)}{\exp\left(-\alpha T\right) N(-d_2^T) - N(-d_1^T)}$$

that is the value of λ for which the expected utility is equal to 0.

The numerator is obviously increasing and converges to 1 as T tends to infinity. For the denominator, things are not so clear; however, we easily get $\lim_{T\to 0} P_T = 0$ and $\lim_{T\to+\infty} P_T = 0$; it then exists at least one point t^* such that $\frac{\partial P_t}{\partial t}\mid_{t^*} = 0$. Consequently, we deduce $\lim_{T\to+\infty} \lambda_T = +\infty$; therefore, for a fixed value of λ, there exists a T^* such that $E(v_{T^*}(S_{T^*}))$ is positive.

To illustrate this point, consider the following parameters :

$$\mu = 0.09; r = 0.04; \sigma = 0.15; t = 50$$

The risk premium, equal to 5%, is a reasonable level with respect to historical data (Mehra-Prescott, 1985 or Kocherlakota, 1996). The dashed line on figure 1 represents the call price when the solid line reports the put price which becomes decreasing beyond the maturity T^*.

FIGURE 1 around here

The price of the put is non-monotonic, thanks to the difference between the expected return on the stock and the risk-free rate. In a risk-neutral world where $\mu = r$, the put and the call prices would be equal, as shown by Bodie (1995); in our framework, it would be interpreted as $\lambda = 1$, a result consistent with our definition of λ.

λ_T, as defined before, tends to infinity with T, meaning that any investor will buy stocks as soon as his investment horizon is sufficiently long (figure 2 for T varying from 0 to 10).

FIGURE 2 around here

However, for short maturities and a fixed λ, the expected utility of the investor is negative. In this case, he will not invest in stocks; figure 3 illustrates this point; we have considered a fixed loss aversion parameter equal to 2.5, a range $[0; 10]$ for the horizon and two values of μ respectively equal to 0.07 and 0.09). The expected utility is negative for horizons less than about one year when $\mu = 0.09$ and for horizons inferior to four years when $\mu = 0.07$.

FIGURE 3 around here

An important point has to be mentioned here. The expected utility may be expressed as :

$$\begin{aligned} E\left(v_T(S_T)\right) &= S_0 \exp((\alpha - r)T) \\ &\times \left[\left(N(d_1^T) - \exp\left(-\alpha T\right) N(d_2^T) \right) - \lambda \left(\exp\left(-\alpha T\right) N(-d_2^T) - N(-d_1^T) \right) \right] \end{aligned}$$

As $\alpha = \mu - r$, we get $\alpha - r = \mu - 2r$; thus, when $\mu < 2r$, the expected utility is not always increasing with respect to maturity for a fixed value of λ, however it remains positive and converges to 0 when T tends to infinity.

This particular shape may be explained by our formulation and the presence of the coefficient $\exp(-rT)$. It was introduced to take into account the fact that gains and losses are not obtained at the same date when T varies. As the utility function is piecewise linear, a gain of one unit has not the same effect, when it is considered at date 0, if it is obtained at date T or at date $T' \neq T$. However, the critical λ is always increasing with maturity because $\exp(-rT)$ is cancelled when the ratio of option prices is evaluated.

It means that more and more investors will invest in stocks when the horizon lengthens.

On a qualitative point of view, it is the same kind of result than the "acceptance property" of Ross (1999) saying that a risk-averse investor may accept a sufficiently long sequence of individually undesirable bets. In our case, the assumption of a geometric brownian motion implies that successive returns are independent and the investor can be viewed as playing successive independent bets.

3.3 Myopic loss aversion and alternative interpretations

The "call to put" ratio, our λ_T, is increasing with T, essentially because the put price becomes decreasing beyond a finite maturity level. This evolution can be explained by the nature of the put considered here. In fact, an agent buying such a put is insured to obtain the reference wealth level at date T. However, he is not insured to preserve his wealth before this horizon. This remark can be linked to the argument of Bernartzi and Thaler (1995) when they assume that investors exhibit myopic loss aversion, that is, consider their expected wealth in the short run (about one year with their data) even if they have a long horizon. Without this type of argument, our preceding results clearly confirm the efficiency of time-diversification and do not allow to explain the equity premium.

Moreover, it is important to notice, first, that the reference wealth level is always written in the form $W_0 \exp(rT)$ with r constant, and second that the volatility parameter is also assumed constant and independent of the horizon under consideration. These two elements were briefly considered in the discrete-time model. Using the analytical formula defining the expected utility, we can now give some comments on these assumptions. If we consider a fixed parameter λ, characterizing the "representative agent", we can get the term structure of interest rates which keeps the agent indifferent to the horizon of investment. The figure 4 represents such a term structure when $\lambda = 2.5,\ \mu = 9\%, \sigma = 20\%$. We obtain capitalization rates varying from 2% to 6.5% for horizons between 1 and 10 years. However, even if this curve has a nice shape in that it looks like a "real" term structure, it must be noticed that the implicit rate tends to $-\infty$ when T tends to 0. This feature comes from the properties of the brownian motion; for very short intervals of length h, the increment of the stochastic component Z_t is $O(\sqrt{h})$, much greater than h.

FIGURE 4 around here

It means that an investor cannot be ready to invest in stocks for the very short run because the risk is not sufficiently compensated by the expected return.

Considering now the volatility argument, we can interpret the equity premium as a return associated to the increasing uncertainty of estimations in the long run. Figure 5 shows for maturities varying from 1 to 3 years the volatility curve implied by a premium of 6%. It can be remarked that the implicit volatility increases very quickly with respect to the horizon. Consequently, it doesn't seem to be a good explanation for the equity premium.

FIGURE 5 around here

4 Concluding remarks

The model presented in this paper is parsimonious because of the assumption of a piecewise utility function. Only two parameters are considered, the loss aversion parameter and the reference wealth level, characterized by a capitalization rate. This last parameter was assumed to be equal to the risk-free rate in the major part of the paper but results are not modified, on a qualitative basis, if an other rate is chosen, as long as it is lower than the expected rate of return on the stock. However, in the last section, we have illustrated the fact that mental accounting or myopic loss aversion is not necessary to justify the equity premium if the reference wealth level is adjusted with an increasing rate. This remark can also be found in Dupré-Louvet (2000) with a direct interpretation of the term structure of interest rates. An other approach can be found in Barberis *et al.*(2001) who consider a reference level which is adjusted through time and among states of nature. In other words the reference level can depend on the history of agent's wealth. More precisely, they assume that agents adapt their reference levels sluggishly. Roughly stated in terms of capitalization rates, agents increase this rate after a better than expected performance. It means that forward short rates are stochastic, decrease on "low wealth trajectories" and increase on "high wealth trajectories".

The other point concerns uncertainty about expectations of future prices. We have illustrated that it is difficult to justify the equity premium by this argument, at least if it used with a constant capitalization rate. Probably, the correct solution consists in the mix of the two approaches.

Our model has two advantages; as said before, the first one is a small number of parameters to be estimated; the second important feature is the simple analytical formulation of the expected utility by means of option pricing theory. However, there is a price to pay for this simplicity, that is th local risk neutrality when the reference level doesn't adjust in a stochastic manner. It implies that we cannot study the problem in a portfolio approach; it is not possible to define the optimal mix between stocks and bonds.

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Figure 1: Option prices versus maturity

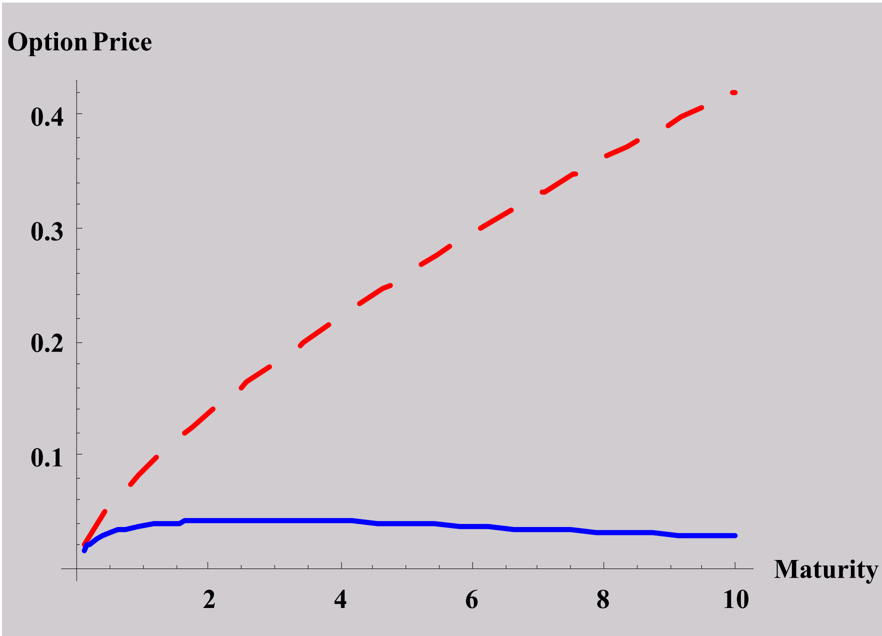

Figure 2: Critical λ versus maturity

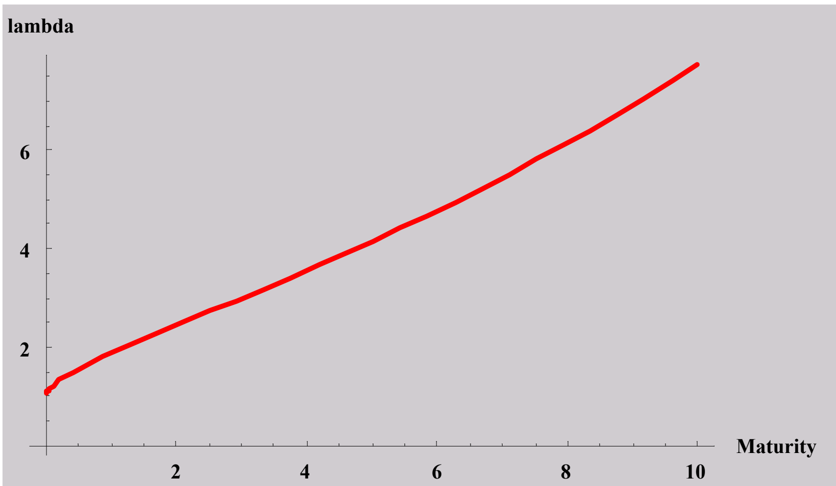

Figure 3: Expected utility versus maturity

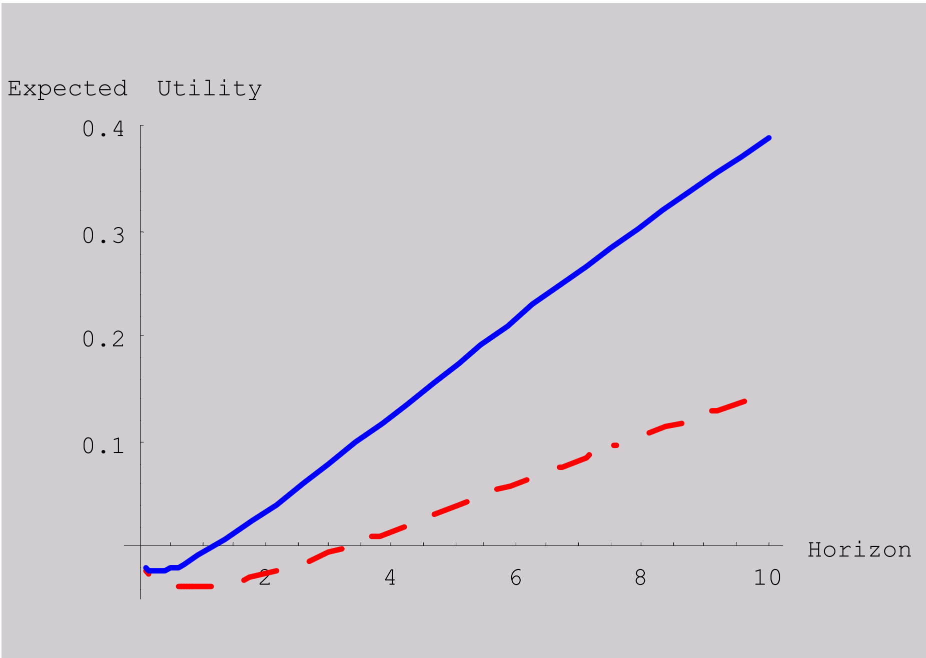

Figure 4: Implied term structure of interest rates

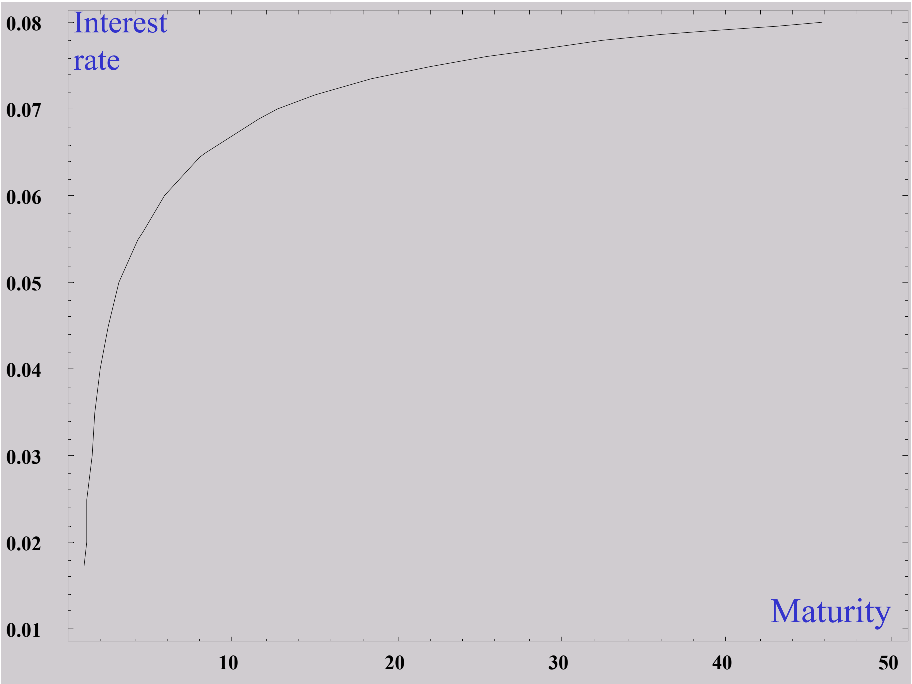

Figure 5: Term structure of volatilities